HOMOLOGY REPRESENTATIONS ARISING FROM THE HALF CUBE, II

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Abstract. In a previous work, we defined a family of subcomplexes of the n-dimensional half cube by removing the interiors of all half cube shaped faces of dimension at least k, and we proved that the reduced homology of such a subcomplex is concentrated in degree $k-1$. This homology group supports a natural action of the Coxeter group $W(D_n)$ of type D. In this paper, we explicitly determine the characters (over $\mathbb{C}$) of these homology representations, which turn out to be multiplicity free. Regarded as representations of the symmetric group $\mathfrak{S}_n$ by restriction, the homology representations turn out to be direct sums of certain representations induced from parabolic subgroups. The latter representations of $\mathfrak{S}_n$ agree (over $\mathbb{C}$) with the representations of $\mathfrak{S}_n$ on the $(k-2)$-nd homology of the complement of the k-equal real hyperplane arrangement.

1. Introduction

The half cube, also known as the demihypercube, may be constructed by selecting one point from each adjacent pair of vertices in the n-dimensional hypercube and taking the convex hull of the resulting set of 2^{n-1} points. In a previous work [**9**], we showed that the k-faces of the half cube $\mathrm{h}\gamma_n$ are of two types: regular simplices and, for $k \geq 3$, isometric copies of half cubes of lower dimension. These faces assemble naturally into a regular CW complex, C_n, which is homeomorphic to a

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ball. Furthermore, for each $3 \leq k \leq n$, there is an interesting subcomplex $C_{n,k}$ of C_n obtained by deleting the interiors of all the half cube shaped faces of dimensions $l \geq k$. We also showed in [**9**, Theorem 3.3.2] that the reduced homology of $C_{n,k}$ is free over $\mathbb{Z}$ and concentrated in degree $k-1$.

The nonzero Betti numbers $B(n,k)$ of $C_{n,k}$ can be characterized by simple recurrence relations: $B(n,0) = B(n,n) = 1$ and, for $0 < k < n$,

$$B(n,k) = 2B(n-1,k) + B(n-1,k-1).$$

There are also nonrecursive formulae for $B(n,k)$; for example, Björner–Welker [**6**, Theorem 1.1 (c)] prove that

$$B(n,k) = \sum_{i=k}^{n} \binom{n}{i} \binom{i-1}{k-1},$$

where we interpret $\binom{-1}{-1}$ to mean 1. The numbers $B(n,k)$ are interesting because they occur in a diverse range of contexts, such as:

(i) in the problem of finding, given n real numbers, a lower bound for the complexity of determining whether some k of them are equal [**4**, **5**, **6**, §1],

(ii) as the $(k-2)$-nd Betti numbers of the k-equal real hyperplane arrangement in $\mathbb{R}^n$ [**6**],

(iii) as the ranks of A-groups appearing in combinatorial homotopy theory [**1**, **2**],

(iv) as the number of nodes used by the Kronrod–Patterson–Smolyak cubature formula in numerical analysis [**15**, Table 3], and

(v) (when $k=3$) in engineering, as the number of three-dimensional block structures associated to n joint systems in the construction of stable underground structures [**12**].

The connections between (i), (ii) and (iii) above are now well understood. Although the half cube polytope has no obvious direct relationship with any of these five phenomena, its associated homology groups share an intriguing feature in common with those appearing in (ii) and (iii): they all support natural actions of the

symmetric group $\mathfrak{S}_n$. One possible way to forge a link between the half cube and the situations in (ii) or (iii) is to try to understand the various homology groups in terms of group representations.

The k-equal real hyperplane arrangement $V^{\mathbb{R}}_{n,k}$ is the set of points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$ for some set of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. The complement $\mathbb{R}^n - V^{\mathbb{R}}_{n,k}$, denoted by $M^{\mathbb{R}}_{n,k}$, is a manifold whose homology is concentrated in degrees $t(k-2)$, where $t \in \mathbb{Z}$ satisfies $0 \leq t \leq \lfloor \frac{n}{k} \rfloor$ (see [**6**, Theorem 1.1(b)]). It is clear from the definition that $M^{\mathbb{R}}_{n,k}$ supports an action of $\mathfrak{S}_n$ via permutation of coordinates, and this action endows the nonzero homology groups with the structure of $\mathfrak{S}_n$-modules. The characters of these modules were computed explicitly by Peeva, Reiner and Welker in [**14**, Theorem 4.4].

The n-dimensional half cube has a large symmetry group G_n of orthogonal transformations acting on it via cellular automorphisms. This group always contains the Coxeter group $W(D_n)$ of order $2^{n-1}n!$, although this containment is proper if $n = 4$, and in turn the group $W(D_n)$ contains a subgroup isomorphic to $\mathfrak{S}_n$. The action of $W(D_n)$ on $\mathrm{h}\gamma_n$ induces, for each k, an action on the nonzero homology groups of $C_{n,k}$. In this paper, we will compute the characters (over $\mathbb{C}$) of these homology representations. Regarded as modules for $W(D_n)$, the representations turn out to be multiplicity free (Theorem 4.4), although they are not generally induced modules in any nontrivial sense. In contrast, if the homology representations are regarded as modules for the symmetric group $\mathfrak{S}_n$ by restriction, then they are no longer multiplicity free, but they do turn out to be isomorphic to direct sums of modules induced from maximal Young subgroups (Theorem 4.7). Furthermore, over the complex numbers, the action of $\mathfrak{S}_n$ on the $(k-1)$-st homology of $C_{n,k}$ agrees with the action of $\mathfrak{S}_n$ on the $(k-2)$-nd homology of $M^{\mathbb{R}}_{n,k}$.

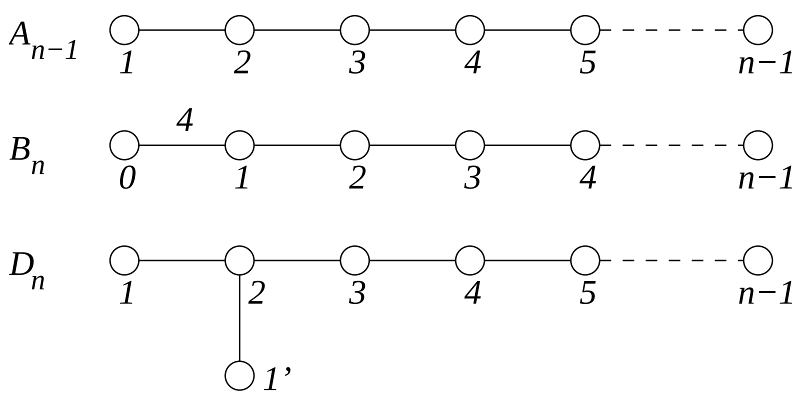
Our results have some interesting combinatorial consequences. One of these (Corollary 4.6) is that if we restrict the representation of $W(D_n)$ on the $(k-1)$-st homology of $C_{n,k}$ to the subgroup $W(D_{n-1})$, then the corresponding branching rule categorifies the usual recurrence relation for the Betti numbers $B(n,k)$. Another

nice property is that if one computes the dimension of the homology representations from a knowledge of their characters, then one obtains a combinatorial proof of the Björner–Welker formula for $B(n, k)$ mentioned above.

2. Character theory of Coxeter groups of classical type

The main groups of interest in this paper are the finite Coxeter groups of classical type, meaning types A_{n-1}, B_n and D_n. It will be convenient to number the vertices of the corresponding Coxeter graphs as shown in Figure 1.

Figure 1. Coxeter graphs of type A_{n-1}, B_n and D_n



We now summarize some well-known properties of these groups. More details may be found in [**11**] or [**3**].

The *Coxeter group* $W = W(\Gamma)$ corresponding to a Coxeter graph Γ with vertices $S = S(\Gamma)$ is given by the presentation

$$\langle s_i : i \in S(\Gamma) : (s_i s_j)^{m_{ij}} = 1 \rangle.$$

The numbers m_{ij} are defined to satisfy $m_{ii} = 1$ and $m_{ij} = m_{ji}$ for all $i, j \in S$. Furthermore, we have $m_{ij} = 2$ if i and j are not adjacent in the graph; $m_{ij} = 3$ if i and j are connected by an unlabelled edge; and $m_{ij} = k$ if i and j are connected by an edge labelled $k > 3$.

If $S' \subset S$, then we refer to the subgroup W' of W that is generated by S' as a *parabolic subgroup* of W. In this case, W' inherits the structure of a Coxeter group from W.

The Coxeter group $W(A_{n-1})$ is isomorphic (as an abstract group) to the symmetric group $\mathfrak{S}_n$, and the Coxeter generator s_i may be identified with the transposition $(i, i+1)$.

The Coxeter group $W(B_n)$ is isomorphic to the wreath product $\mathbb{Z}_2 \wr \mathfrak{S}_n$ of order $2^n n!$. This may be regarded as a group of permutations of n signed objects, in which s_i acts by the transposition $(i, i+1)$ for $1 \leq i < n$, and s_0 acts by changing the sign of the object numbered 1. The parabolic subgroup of $W(B_n)$ obtained by omitting the generator s_0 is canonically isomorphic (as a Coxeter group) to $W(A_{n-1})$.

The Coxeter group $W(D_n)$, which will be our main group of interest, can be identified with the index 2 subgroup of $W(B_n)$ consisting of those elements effecting an even number of sign changes. As before, we may identify s_i with the permutation $(i, i+1)$ for $1 \leq i < n$. The other generator, $s_{1'}$, can be identified with the element $s_0 s_1 s_0$ of $W(B_n)$. It therefore acts by changing the sign of each of objects 1 and 2, followed by the transposition $(1, 2)$. The parabolic subgroup of $W(D_n)$ obtained by omitting the generator $s_{1'}$ is canonically isomorphic (as a Coxeter group) to $W(A_{n-1})$. We will often abuse notation slightly and refer to this subgroup as $\mathfrak{S}_n$.

For any Coxeter group (W, S), there is a unique homomorphism $\varepsilon : W \longrightarrow \{\pm 1\}$ to the multiplicative group of 2 elements sending each element of S to -1. This homomorphism is known as the *sign representation.* We will write sgn_n (respectively, id_n) to denote the sign (respectively, trivial) representation of any of the Coxeter groups of types A_{n-1}, B_n or D_n.

The character theory of finite Coxeter groups of classical type is well understood. It is described in Geck and Pfeiffer's book [**7**, §5] and, more explicitly, in Stembridge's notes on the topic [**17**]. We now summarize some of the key properties of the theory for later use.

The irreducible representations of $W(A_{n-1})$ (over $\mathbb{C}$) are indexed by the partitions of n, or equivalently, the set of Young diagrams of size n. We will write the corresponding set of characters as

$$\{\chi^\lambda : |\lambda| = n\}.$$

The degree, $\chi^\lambda(1)$, of χ^λ is the number of standard Young tableaux of shape λ; that is, the number of ways of filling a Young diagram of shape λ with the numbers $1, 2, \ldots, n$ once each in such a way that the entries increase along rows and down columns. The identity character corresponds to the partition $\lambda = [n]$, whose Young diagram has one row, and the sign character corresponds to the partition $\lambda = [1^n]$, whose Young diagram has one column.

Another important character for our purposes corresponds to the partition

$$[n-1, 1],$$

which gives the character of the *reflection representation* associated to the Coxeter group of type A_{n-1}. This representation may also be constructed by first taking the n-dimensional representation of $\mathfrak{S}_n$ corresponding to the natural action of the group on n letters, and then quotienting by the 1-dimensional submodule spanned by the all-ones vector.

The irreducible characters of $W(B_n)$ are indexed by the set

$$\{\chi^{(\mu,\nu)} : |\mu| + |\nu| = n\}.$$

The dimensions of the irreducibles may be obtained from the corresponding dimensions in type A via the formula

$$\chi^{(\mu,\nu)}(1) = \binom{n}{|\mu|} \chi^\mu(1)\chi^\nu(1).$$

The identity character corresponds to the pair $([n], [0])$, and the sign character to the pair $([0], [1^n])$.

As described above, we may regard $W(D_n)$ as a subgroup of $W(B_n)$. Under this identification, the irreducible characters $\chi^{(\mu,\nu)}$ and $\chi^{(\nu,\mu)}$ (where $\mu \neq \nu$) both restrict to the same irreducible character of $W(D_n)$, which we denote by $\chi^{\{\mu,\nu\}}$. On the other hand, the irreducible character $\chi^{(\mu,\mu)}$ of $W(B_n)$ restricts to a sum of two nonisomorphic irreducible characters of $W(D_n)$ of the same degree; we denote the

latter by $\chi_+^{\{\mu,\mu\}}$ and $\chi_-^{\{\mu,\mu\}}$. These exhaust all the irreducible characters of $W(D_n)$. In other words, the irreducible characters of $W(D_n)$ are indexed by the set

$$\{\chi^{\{\mu,\nu\}} : |\mu| + |\nu| = n\} \cup \{\chi_\pm^{\{\mu,\mu\}} : |\mu| = n/2\},$$

where the second subset is empty if n is odd. The identity character corresponds to the pair $\{[n], [0]\}$ and the sign character corresponds to the pair $\{[1^n], [0]\}$. It is immediate from the above remarks that the dimensions of the corresponding irreducibles are given by

$$\chi^{\{\mu,\nu\}}(1) = \binom{n}{|\mu|}\chi^\mu(1)\chi^\nu(1)$$

and

$$\chi_\pm^{\{\mu,\mu\}}(1) = \frac{1}{2}\binom{n}{|\mu|}\chi^\mu(1)^2.$$

The following two lemmas concerning characters of $W(D_n)$ will be important in the sequel. It will sometimes be convenient to write $\chi_\varepsilon^{\{\mu,\nu\}}$ to refer to the irreducible character $\chi^{\{\mu,\nu\}}$ if $\mu \neq \nu$, and to refer to either of the irreducible characters $\chi_+^{\{\mu,\nu\}}$ or $\chi_-^{\{\mu,\nu\}}$ if $\mu = \nu$.

Lemma 2.1. *Let $\mathfrak{S}_n$ be the parabolic subgroup of $G = W(D_n)$ obtained by omitting the generator $s_{1'}$. Let $m = \lfloor \frac{n}{2} \rfloor$.*

(i) *If $\mu \neq \nu$, then we have*

$$\chi^{\{\mu,\nu\}} \downarrow_{\mathfrak{S}_n}^G = \sum_\lambda c_{\mu\nu}^\lambda \chi^\lambda,$$

where the $c_{\mu\nu}^\lambda$ are the Littlewood–Richardson coefficients.

(ii) *If n is odd, then we have*

$$\mathrm{id}_n \uparrow_{\mathfrak{S}_n}^G = \sum_{l \leq m} \chi^{\{[l],[n-l]\}}.$$

(iii) *If n is even, then we have*

$$\mathrm{id}_n \uparrow_{\mathfrak{S}_n}^G = \chi_+^{\{[m],[m]\}} + \sum_{l < m} \chi^{\{[l],[n-l]\}}.$$

Proof. Part (i) appears in [**17**, §3A]. Under the hypotheses of part (ii), we must have $\mu \neq \nu$ because n is odd. The conclusion of (ii) then follows from (i) by using Frobenius reciprocity and the Pieri Rule. Part (iii) appears in [**17**, §3C]. □

Lemma 2.2. *Let $G = W(D_n)$, let $3 \leq k \leq n$, and let D_k (respectively, D_{n-k}) be the parabolic subgroup of G generated by the set*

$$\{s_{1'}\} \cup \{s_i : 1 \leq i < k\}$$

(respectively, $\{s_i : i > k\}$.) Denote the usual inner product on characters by $\langle , \rangle$. Suppose below that the ordered pairs (α, ψ) and (β, θ) are not equal.

(i) *If $\mu \neq \nu$ then we have*

$$\left\langle \chi^{\{\mu,\nu\}} \downarrow^G_{D_k \times D_{n-k}},\ \chi_\varepsilon^{\{\alpha,\beta\}} \times \chi_{\varepsilon'}^{\{\psi,\theta\}} \right\rangle = c^\mu_{\alpha\psi} c^\nu_{\beta\theta} + c^\mu_{\alpha\theta} c^\nu_{\beta\psi} + c^\mu_{\beta\psi} c^\nu_{\alpha\theta} + c^\mu_{\beta\theta} c^\nu_{\alpha\psi}.$$

(ii) *We have*

$$\left\langle \chi_\pm^{\{\mu,\mu\}} \downarrow^G_{D_k \times D_{n-k}},\ \chi_\varepsilon^{\{\alpha,\beta\}} \times \chi_{\varepsilon'}^{\{\psi,\theta\}} \right\rangle = c^\mu_{\alpha\psi} c^\mu_{\beta\theta} + c^\mu_{\alpha\theta} c^\mu_{\beta\psi}.$$

Proof. After applying Frobenius reciprocity, this becomes a restatement of results in [**17**, §3A, §3D]. □

The irreducible characters in type B_n have the following well-known branching rule, which will be useful in the sequel.

Lemma 2.3. *Let $\chi^{(\lambda,\mu)}$ be an irreducible character for $W(B_n)$. Then we have*

$$\chi^{(\lambda,\mu)} \downarrow^{W(B_n)}_{W(B_{n-1})} = \sum_{d \in I(\lambda)} \chi^{(\lambda^{(d)},\mu)} + \sum_{d \in I(\mu)} \chi^{(\lambda,\mu^{(d)})},$$

where $I(\lambda)$ is the set of removable boxes in the Young diagram corresponding to λ, and $\lambda^{(d)}$ is the result of removing box d from the Young diagram.

Proof. This appears in [**7**, §6.1.9]. □

Corollary 2.4. *Maintain the notation of Lemma 2.3. Suppose in addition that each character $\chi^{(\alpha,\beta)}$ appearing in Lemma 2.3 satisfies $\alpha \neq \beta$. Then we have*

$$\chi^{\{\lambda,\mu\}} \downarrow^{W(D_n)}_{W(D_{n-1})} = \sum_{d \in I(\lambda)} \chi^{\{\lambda^{(d)},\mu\}} + \sum_{d \in I(\mu)} \chi^{\{\lambda,\mu^{(d)}\}}.$$

Proof. Recall that each type B character $\chi^{(\alpha,\beta)}$ appearing in the statement restricts to the irreducible type D character $\chi^{\{\alpha,\beta\}}$. The result now follows from Lemma 2.3 and the fact that, under the usual identifications, we have $W(B_{n-1}) \cap W(D_n) = W(D_{n-1})$. $\square$

3. The half cube

An n-dimensional *(Euclidean) polytope* Π_n is a closed, bounded, convex subset of $\mathbb{R}^n$ enclosed by a finite number of hyperplanes. The part of Π_n that lies in one of the hyperplanes is called a *facet*, and each facet is an $(n-1)$-dimensional polytope. A polytope is homeomorphic to an n-ball (which follows, for example, from [**13**, Lemma 1.1]), and the boundary of the polytope, which is equal to the union of its facets, is identified with the $(n-1)$-sphere by this homeomorphism.

Iterating this construction gives rise to a set of k-dimensional polytopes Π_k (called *k-faces*) for each $0 \leq k \leq n$. The elements of Π_0 are called *vertices* and the elements of Π_1 are called *edges*. It is not hard to show that a polytope is the convex hull of its set of vertices, and that the boundary of a polytope is precisely the union of its k-faces for $0 \leq k < n$. What is less obvious, but still true [**18**, Theorem 1.1], is that the convex hull of an arbitrary finite subset of $\mathbb{R}^n$ is a polytope in the above sense. It follows that a polytope is determined by its vertex set, and we write $\Pi(V)$ for the polytope whose vertex set is V.

Definition 3.1. Let $n \geq 4$ be an integer, and let $\mathbf{n} = \{1, 2, \ldots, n\}$.

We define the set Ψ_n to be the set of 2^n vertices of the hypercube whose coordinates are

$$(\pm 1, \pm 1, \ldots, \pm 1).$$

Let Ψ_n^+ be the set of 2^{n-1} vertices with an even number of negative coordinates, and let Ψ_n^- be $\Psi_n \backslash \Psi_n^+$.

Let $\mathbf{v}' \in \Psi_n^-$ and $S \subseteq \mathbf{n}$. We define the subset $K(\mathbf{v}', S)$ of Ψ_n^+ by the condition that $\mathbf{v} \in K(\mathbf{v}', S)$ if and only if $\mathbf{v}$ and $\mathbf{v}'$ differ only in the i-th coordinate, and $i \in S$.

Let $\mathbf{v} \in \Psi_n^+$ and let $S \subseteq \mathbf{n}$. We define the subset $L(\mathbf{v}, S)$ of Ψ_n^+ by the condition that $\mathbf{v}' \in L(\mathbf{v}, S)$ if and only if $\mathbf{v}$ and $\mathbf{v}'$ agree in the i-th coordinate whenever $i \notin S$. The set S is characterized as the set of coordinates at which not all points of $L(\mathbf{v}, S)$ agree.

The k-faces of the half cube were classified in [**9**].

Theorem 3.2 ([9]). *The k-faces of $\mathrm{h}\gamma_n$ for $k \leq n$ are as follows:*

(i) 2^{n-1} *0-faces (vertices) given by the elements of Ψ_n^+;*

(ii) $2^{n-2}\binom{n}{2}$ *1-faces $\Pi(K(\mathbf{v}', S))$, where $\mathbf{v}' \in \Psi_n^-$ and $|S| = 2$;*

(iii) $2^{n-1}\binom{n}{3}$ *simplex shaped 2-faces $\Pi(K(\mathbf{v}', S))$, where $\mathbf{v}' \in \Psi_n^-$ and $|S| = 3$;*

(iv) $2^{n-1}\binom{n}{k+1}$ *simplex shaped k-faces $\Pi(K(\mathbf{v}', S))$, where $\mathbf{v}' \in \Psi_n^-$ and $|S| = k+1$ for $3 \leq k < n$;*

(v) $2^{n-k}\binom{n}{k}$ *half cube shaped k-faces $\Pi(L(\mathbf{v}, S))$, where $\mathbf{v} \in \Psi_n^+$ and $|S| = k$ for $3 \leq k \leq n$.*

Furthermore, two faces are conjugate under the action of $W(D_n)$ if and only if they have the same dimension and the same shape.

Proof. The classification of the k-faces is given in [**9**, Theorem 2.3.6], and the classification of the orbits under the action of $W(D_n)$ is given in [**9**, Theorem 4.2.3 (ii)]. $\square$

The unique n-face in (v) above corresponds to the interior of the polytope. The k-faces assemble naturally into a regular CW complex, C_n.

Definition 3.3. For each integer k with $-1 \leq k \leq n$, let V_k be the k-th chain group in the complex C_n. For $k \geq 3$, we write $V_k = X_k \oplus Y_k$, where X_k (respectively,

Y_k) is the span of the simplex-shaped (respectively, half-cube shaped) faces. If $-1 \leq k < 3$, we define $X_k = V_k$ and $Y_k = 0$.

We now recall some of the key properties of this complex; the reader is referred to [**9**] for full details.

For any fixed k such that $3 \leq k \leq n$, one may form a CW subcomplex $C_{n,k}$ by removing the interiors of all the half cube shaped l-faces for $l \geq k$. In other words, the l-th chain group of $C_{n,k}$ is equal to V_k if $l < k$, and to X_k if $l \geq k$. The reduced (cellular) homology of $C_{n,k}$ is free over $\mathbb{Z}$ and concentrated in degree $k-1$ [**9**, Theorem 3.3.2].

The Coxeter group $W(D_n)$ acts naturally on Ψ_n^+ (and also on Ψ_n^-) via signed permutations of the coordinates. This induces an action of $W(D_n)$ on the half cube $\mathrm{h}\gamma_n$ via cellular automorphisms. In particular, elements of $W(D_n)$ send k-faces of $\mathrm{h}\gamma_n$ to other k-faces of the same type (i.e., simplex shaped or half cube shaped), which means that the free $\mathbb{Z}$-modules X_k and Y_k of Definition 3.3 acquire the structure of $W(D_n)$-modules. In turn, there is an induced action of $W(D_n)$ on the subcomplex $C_{n,k}$ via cellular automorphisms, as well as on the homology groups of $C_{n,k}$ [**9**, Theorem 4.2.3].

The following basic result will be of key importance in the sequel.

Lemma 3.4. *Let $n \geq 3$ and let s be a Coxeter generator of the group $G = W(D_n)$. The element s acts on the half cube $\mathrm{h}\gamma_n$ by a reflection in a hyperplane through the origin. The induced action of s on $H_{n-1}(C_{n,n})$ and on the n-th chain group of C_n is negation.*

Proof. The first assertion follows from [**8**, Proposition 3.6, Lemma 5.3].

The CW space corresponding to the subcomplex $C_{n,n}$ is obtained from C_n by deleting the (interior of the) unique n-cell. It follows that this space is homeomorphic to S^{n-1}. A well-known result [**10**, 2.2 (e)] then shows that s acts on $H_{n-1}(S^{n-1}; \mathbb{Z})$ by negation, proving the first part of the second assertion. For the final assertion, we use the fact that s acts continuously on $\mathrm{h}\gamma_n$, which means that

it acts on the chain complex of C_n by a chain map. Since the n-th chain group of C_n has rank 1, the fact that s acts by negation on $H_{n-1}(S^{n-1};\mathbb{Z})$ forces it to act by negation on C_n, which completes the proof. $\square$

Lemma 3.5. *Let $n \geq 4$, $G = W(D_n)$ and let k satisfy $3 \leq k \leq n$. Let D_k denote the parabolic subgroup of $W(D_n)$ generated by the set*

$$\{s_{1'}\} \cup \{s_i : 1 \leq i < k\},$$

and let $\mathfrak{S}_{n-k}$ denote the parabolic subgroup of $W(D_n)$ generated by the set

$$\{s_i : i > k\}.$$

Regarding Y_k as a $\mathbb{C}G$-module by extension of scalars, we have

$$Y_k \cong_G (\operatorname{sgn}_k \otimes \operatorname{id}_{n-k}) \uparrow_{D_k \times \mathfrak{S}_{n-k}}^G .$$

Proof. By Theorem 3.2, there is one orbit of half cube shaped faces for each $3 \leq k \leq n$. One of these has vertex set $L(\mathbf{v}, S)$, where

$$\mathbf{v} = (1, 1, \ldots, 1)$$

and

$$S = \{k+1, k+2, \ldots, n\}.$$

Let e be the k-cell of the CW complex C_n corresponding to $L(\mathbf{v}, S)$. It is clear from the definitions of the action of G as signed permutations that the set $L(\mathbf{v}, S)$ is fixed setwise by all the s_i other than s_k. The group generated by this subset of the generators is $G_k := D_k \times \mathfrak{S}_{n-k}$, which has order $2^{k-1}k!(n-k)!$ and index

$$2^{n-k}\binom{n}{k}$$

in G. It now follows from Theorem 3.2 (v) that G_k is the full set stabilizer of $L(\mathbf{v}, S)$.

The Coxeter generators $s_{1'}$, $s_1, \ldots, s_{k-1}$ of D_k act as reflections in hyperplanes through the origin. Lemma 3.4 then shows that each of these generators sends e to $-e$. In contrast, the Coxeter generators $s_{k+1}, s_{k+2}, \ldots, s_n$ fix $L(\mathbf{v}, S)$ (and its convex hull) pointwise. These generators fix e.

The assertion follows from the above observations. □

Lemma 3.6. *Maintain the notation of Lemma 3.5, and let $\eta(k,e)$ denote the partition $[e+1, 1, \ldots, 1]$ of $k+e$. The character of the module Y_k is given by*

$$\sum_{e \leq n-k} \chi^{\{\eta(k,e),[n-k-e]\}} + \sum_{e' \leq n-(k+1)} \chi^{\{\eta(k+1,e'),[n-(k+1)-e']\}}.$$

Proof. By transitivity of induction and Lemma 3.5, we have

$$Y_k \cong_G \left((\operatorname{sgn}_k \otimes \operatorname{id}_{n-k}) \uparrow_{D_k \times \mathfrak{S}_{n-k}}^{D_k \times D_{n-k}} \right) \uparrow_{D_k \times D_{n-k}}^{G} .$$

Let $m = \lfloor \frac{n-k}{2} \rfloor$. By Lemma 2.1, we have

$$(\operatorname{sgn}_k \otimes \operatorname{id}_{n-k}) \uparrow_{D_k \times \mathfrak{S}_{n-k}}^{D_k \times D_{n-k}} = \sum_{l \leq m} \chi^{\{[1^k],[0]\}} \times \chi^{\{[l],[n-k-l]\}}$$

if $n-k$ is odd, and

$$(\operatorname{sgn}_k \otimes \operatorname{id}_{n-k}) \uparrow_{D_k \times \mathfrak{S}_{n-k}}^{D_k \times D_{n-k}} = \chi^{\{[1^k],[0]\}} \times \chi_+^{\{[m],[m]\}} + \sum_{l < m} \chi^{\{[1^k],[0]\}} \times \chi^{\{[l],[n-k-l]\}}$$

if $n-k$ is even. Lemma 2.2 is applicable in this situation, because neither of the partitions $[1^k]$ or $[0]$ has one row. The assertion now follows from Lemma 2.2 and the Pieri rule. (Observe that the numbers appearing in Lemma 2.2 (ii) are always zero in this case.) □

Remark 3.7. The methods used in Lemma 3.6 to determine the characters of the modules Y_k can be extended to compute the characters of the modules X_k, as well as the characters of all the representations corresponding to cycles and to boundaries in the subcomplexes $C_{n,k}$.

4. Main results

In order to prove our main results, we require a version of the Hopf trace formula that applies in contexts more general than simplicial complexes.

Theorem 4.1 (Hopf trace formula [13, Theorem 22.1]). *Let K be a finite complex with (integral) chain groups $C_p(K)$ and homology groups $H_p(K)$. Let $T_p(K)$ be the torsion subgroup of $H_p(K)$. Let $\phi : C_p(K) \longrightarrow C_p(K)$ be a chain map, and let ϕ_* be the induced map on homology. Then we have*

$$\sum_p (-1)^p \operatorname{tr}(\phi, C_p(K)) = \sum_p (-1)^p \operatorname{tr}(\phi_*, H_p(K)/T_p(K)).$$

□

Lemma 4.2. *Consider the CW complex C_n of Definition 3.3; its chain groups are the V_l for $-1 \leq l \leq n$. Let ϕ be a chain map of this chain complex. Then we have*

$$\sum_p (-1)^p \operatorname{tr}(\phi, V_p) = 0.$$

Proof. The chain complex C_n is a CW decomposition of the half cube, which is a contractible space and has trivial reduced homology. Theorem 4.1 applies to the complex C_n, and the previous observation shows that the right hand side of the Hopf trace formula is zero, completing the proof. □

Lemma 4.3. *Consider the CW subcomplex $C_{n,k}$ of C_n; its chain groups are V_l for $-1 \leq l < k$ and X_l for $k \leq l \leq n$, where $V_l = X_l \oplus Y_l$ for $l \geq 3$. Let ϕ be a chain map of this chain complex. Then we have*

$$\operatorname{tr}(\phi_*, H_{k-1}(C_{n,k})) = \sum_{l \geq k} (-1)^{l-k} \operatorname{tr}(\phi, Y_l).$$

Proof. We first apply the Hopf trace formula to $C_{n,k}$ to obtain

$$\sum_p (-1)^p \operatorname{tr}(\phi, C_p(C_{n,k})) = \sum_p (-1)^p \operatorname{tr}(\phi_*, H_p(C_{n,k}));$$

there is no torsion because the homology of $C_{n,k}$ is free over $\mathbb{Z}$ by [**9**, Theorem 3.3.2]. Since, by the same result, the reduced homology of $C_{n,k}$ is concentrated in degree $k-1$, this simplifies to

$$(-1)^{k-1} \operatorname{tr}(\phi_*, H_{k-1}(C_{n,k})) = \sum_p (-1)^p \operatorname{tr}(\phi, C_p(C_{n,k})).$$

By Lemma 4.2, we have

$$\sum_p (-1)^p \operatorname{tr}(\phi, C_p(C_{n,k})) + \sum_{p \geq k} (-1)^p \operatorname{tr}(\phi, Y_k) = 0,$$

which, combined with the preceding equation, gives

$$(-1)^{k-1} \operatorname{tr}(\phi_*, H_{k-1}(C_{n,k})) = \sum_{p \geq k} (-1)^{p+1} \operatorname{tr}(\phi, Y_k) = 0.$$

The assertion now follows by multiplying both sides by $(-1)^{k-1}$. $\square$

Theorem 4.4. *Let $n \geq 4$, $G = W(D_n)$ and let k satisfy $3 \leq k \leq n$. Let $\eta(k,e)$ denote the partition $[e+1, 1, \ldots, 1]$ of $k+e$. The character of the representation of G on the $(k-1)$-st homology of the complex $C_{n,k}$ is given by*

$$\chi_D(n,k) = \sum_{e \leq n-k} \chi^{\{\eta(k,e), [n-k-e]\}}.$$

Proof. Let χ_k denote the character of the G-module Y_k. By Lemma 4.3, the character of the homology representation is given by the alternating sum

$$\chi_k - \chi_{k+1} + \chi_{k+2} - \chi_{k+3} \cdots .$$

The result now follows from Lemma 3.6: all of the terms appearing in the statement of that result cancel, except those involving a partition of the form $\eta(l,e)$ for $l = k$. $\square$

Remark 4.5. It is known [**16**, §4] that the k-th exterior power of the (n-dimensional) reflection representation of $W(D_n)$ is irreducible and corresponds to the pair of partitions

$$\{[1^k], [n-k]\} = \{\eta(k,0), [n-k-0]\}.$$

Theorem 4.4 shows that this is one of the constituents of the representation of $W(D_n)$ on the $(k-1)$-st homology of $C_{n,k}$.

Corollary 4.6. *Maintain the notation of Theorem 4.4, and assume that $k < n$. We have*

$$\chi_D(n,k) \downarrow^{W(D_n)}_{W(D_{n-1})} = 2\chi_D(n-1,k) + \chi_D(n-1,k-1).$$

Proof. Since $k \geq 3$, Corollary 2.4 shows that

$$\begin{aligned}\chi^{\{\eta(k,e),[n-k-e]\}} \downarrow^{W(D_n)}_{W(D_{n-1})} =& \chi^{\{\eta(k,e-1),[n-k-e]\}} + \chi^{\{\eta(k,e),[n-k-e-1]\}} \\ & + \chi^{\{\eta(k-1,e),[n-k-e]\}} \\ =& \chi^{\{\eta(k,e-1),[(n-1)-k-(e-1)]\}} + \chi^{\{\eta(k,e),[(n-1)-k-e]\}} \\ & + \chi^{\{\eta(k-1,e),[(n-1)-(k-1)-e]\}},\end{aligned}$$

where we ignore any terms involving partitions with negative parts. The result now follows by summing over e, as in Theorem 4.4. $\square$

Theorem 4.7. *Let $n \geq 4$ and let k satisfy $3 \leq k \leq n$. Let $\mathfrak{S}_n$ denote the parabolic subgroup of $W(D_n)$ corresponding to the omission of the generator $s_{1'}$. Let E_l be the $(l-1)$-dimensional reflection representation for $\mathfrak{S}_l$ described in §2.*

(i) *Regarded as a $\mathbb{C}\mathfrak{S}_n$-module by restriction, the $(k-1)$-st homology of the complex $C_{n,k}$ is isomorphic to*

$$\bigoplus_{e \leq n-k} \left(\mathrm{id}_{n-k-e} \otimes \bigwedge^{k-1} E_{k+e}\right) \uparrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-k-e} \times \mathfrak{S}_{k+e}} .$$

(ii) *As $\mathbb{C}\mathfrak{S}_n$-modules, the $(k-1)$-st homology of $C_{n,k}$ is isomorphic to the $(k-2)$-nd (co)homology of the complement, $M^{\mathbb{R}}_{n,k}$, of the k-equal real hyperplane arrangement.*

Proof. We first prove (i). By Theorem 4.4, it is enough to show that, for $e \leq n-k$, the restriction of the character $\chi^{\{\eta(k,e),[n-k-e]\}}$ of $W(D_n)$ to $\mathfrak{S}_n$ corresponds to the representation

$$\left(\mathrm{id}_{n-k-e} \otimes \bigwedge^{k-1} E_{k+e}\right) \uparrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-k-e} \times \mathfrak{S}_{k+e}}$$

of $\mathfrak{S}_n$.

By [**7**, Proposition 5.4.12], the character of the $\mathfrak{S}_{k+e}$ module $\bigwedge^{k-1} E_{k+e}$ is given by the partition $\mu = [e+1, 1^{k-1}]$. The character of the $\mathfrak{S}_{n-k-e}$-module id_{n-k-e} is given by the one-row partition $\nu = [n-k-e]$. Using standard results [**7**, Definition 6.1.1], the character of the induction product of these two characters to $\mathfrak{S}_n$ is

$$\sum_\lambda c^\lambda_{\mu\nu} \chi^\lambda.$$

The proof of (i) is completed by Lemma 2.1 (i), which shows that we also have

$$\chi^{\{\eta(k,e),[n-k-e]\}} \downarrow^{W(D_n)}_{\mathfrak{S}_n} = \chi^{\{\mu,\nu\}} \downarrow^{W(D_n)}_{\mathfrak{S}_n} = \sum_\lambda c^\lambda_{\mu\nu} \chi^\lambda.$$

Substituting $s = 1$ into [**14**, Theorem 4.4], we see that the complex character of the $(k-2)$-nd cohomology of $M^{\mathbb{R}}_{n,k}$, regarded as a $\mathbb{C}\mathfrak{S}_n$-module, agrees with the character of the representation described in part (i). This completes the proof of (ii). □

Remark 4.8. Note that, under the usual identifications, we have

$$W(D_{n-1}) \cap W(A_n) = W(A_{n-1}).$$

It follows that the type A homology representations described in Theorem 4.7 have a branching rule analogous to the type D branching rule of Corollary 4.6. However, this would not be such an obvious result in the absence of the wider context of the type D representations.

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